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Baskoro, Edy T. (IN-BITNS-CBM); **Messegué, Arnau**; **Miret, Josep M.** (E-LLE)

New results regarding the permutation cycle structure of almost Moore digraphs. (English. English summary)

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The degree/diameter problem asks for the maximum order of a graph with maximum degree d and diameter k. The *Moore bound* refers to a natural upper bound for the order of such a graph, and graphs that attain this bound are called *Moore graphs*. An important open problem is whether there exists a Moore graph of diameter $k=2$ and maximum degree $d=57$. For a dynamic survey of the area, see [M. Miller and J. Širáň, Electron. J. Combin. **DS14** (2005), Dynamic Surveys, 61 pp.; MR4336216].

The degree/diameter problem has also been studied for directed graphs with maximum outdegree d and diameter k, for which the Moore bound is $M_{d,k}=1+d+d^2+\cdots+d^k$. It turns out that Moore digraphs do not exist unless $d=1$ or $k=1$. Then, a natural problem is to investigate *almost Moore digraphs*, also known as (d,k)*-digraphs of defect 1*, which are digraphs of maximum outdegree d and diameter k whose order is one less than the Moore bound. Almost Moore digraphs have been shown to exist for $k=2$ [see M. A. Fiol, I. Alegre, J. L. A. Yebra, ACM SIGARCH Comput. Arch. News **11** (1983), no. 3, 174–177, doi:10.1145/1067651.801653; J. Gimbert, Discrete Math. **231** (2001), no. 1-3, 177–190; MR1821958]. It is believed that almost Moore digraphs do not exist for $k\geq 3$, and there are several partial results in the literature that support this conjecture. For instance, Miller and I. Friš [in *Graphs, matrices, and designs*, 269–278, Lecture Notes in Pure and Appl. Math., 139, Dekker, New York, 1993; MR1209196] showed the nonexistence for $d=2$, E. T. Baskoro et al. [J. Graph Theory **48** (2005), no. 2, 112–126; MR2110581] showed the nonexistence for $d=3$, and J. Conde et al. showed the nonexistence for $k=3$ [Electron. J. Combin. **15** (2008), no. 1, Research Paper 87; MR2426150] and $k=4$ [Electron. J. Combin. **20** (2013), no. 1, Paper 75; MR3040637].

In this article, the authors show that in order to prove the nonexistence of almost Moore digraphs, it suffices to restrict attention to those with the following specific structure. It is known that in any (d,k)-digraph G of defect 1, for each $v\in V(G)$ there is a unique vertex $r(v)\in V(G)$, called the *repeat* of v, such that there are exactly two $v\to r(v)$ walks in G of length at most k. The map $r\colon V(G)\to V(G)$ turns out to be a permutation of $V(G)$ (and also an automorphism of G), as shown by Baskoro et al. [J. Graph Theory **20** (1995), no. 3, 339–349; MR1355433]. Let m_i be the number of cycles of length i in the cycle decomposition of r as a permutation of $V(G)$, and call the sequence $(m_0,m_1,m_2,\dots)$ the "permutation cycle structure" of G. Then, the authors prove that any almost Moore digraph G with diameter $k>2$ has an induced subgraph H that is an almost Moore digraph of order N whose permutation cycle structure is $(k,0,\dots,0,m_s,0,\dots)$ where $N=k+sm_s$, or $(0,\dots,0,m_s,0,\dots)$ where $N=sm_s$, or $(0,\dots,0,m_s,0,\dots,0,m_{2s},0,\dots)$ where $N=sm_s+2sm_{2s}$.

The subgraph H is induced on the set S of vertices whose order divides ℓ, where $\ell=s$ or $2s$ as the case may be; here, the *order* of a vertex v is defined to be the least integer $t>0$ such that $r^t(v)=v$. To prove the results, the authors work with a block decomposition of the adjacency matrix of G with blocks defined by S and $V(G)\setminus S$. The results in this paper also generalize the results of A. A. Sillasen [Electron. J. Graph Theory Appl. (EJGTA) **3** (2015), no. 1, 1–7; MR3333722] for $d=4$ and $d=5$.

Brahadeesh Sankarnarayanan